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$\mathcal{N}=2$ Supersymmetric Gauge Theories, Branes and Orientifolds

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Abstract

Starting with configurations of fourbranes, fivebranes, sixbranes and orientifolds in Type IIA string theory we derive via M-theory the curves solving $\mathcal{N}=2$ supersymmetric gauge theories with gauge groups $SO(N)$ and $Sp(2N)$. We also obtain new curves describing theories with product gauge groups. A crucial role in the discussion is played by the interaction of the orientifolds with the NS-fivebranes.

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1. Introduction

Recently it has become increasingly clear that D(irichlet)-branes [1] are an extremely powerful tool for studying supersymmetric gauge theories. In fact many phenomena are actually better understood from the D-brane point of view. There are essentially two philosophies for studying gauge theories with D-branes. One is to compactify string theory on a Calabi-Yau space. The BPS-states of the gauge theory can then be identified as the wrapping modes of certain branes around the non-trivial homology cycles of the Calabi-Yau space [2]. A recent review of this geometrical engineering [3] of gauge theories can be found in [4]. A different approach has been pioneered in [5] and further developed in [6–13] and especially in [14]. Here one utilizes the fact that branes themselves can end on branes [15] to construct gauge theories from intersecting branes in a flat background spacetime. More precisely, D-threebranes in Type IIB or D-fourbranes in Type IIA string theory can stretch between a pair of NS fivebranes. Since the D-branes are finite in one direction the effective world volume gauge theory will be three-dimensional in the IIB case or four-dimensional in the IIA case. Using the $SL(2, Z)$ symmetry of IIB string theory one can then rederive [5,6] the recently discovered mirror symmetry of three-dimensional supersymmetric gauge theories [16]. On the Type IIA side the dualities [17] which are characteristic for $\mathcal{N} = 1$ supersymmetric gauge theories in four dimensions¹ have been recovered [7–13].

$\mathcal{N} = 2$ supersymmetric gauge theories in four dimensions were the subject of interest in [14]. The Coulomb branch of the moduli space of four-dimensional $\mathcal{N} = 2$ gauge theories coincides with the moduli space of a particular family of Riemann surfaces. This was first shown in the ground-breaking work of Seiberg and Witten [19] for the case of the gauge group $SU(2)$. In a series of papers this has been generalized to the other classical gauge groups [20–27]². The construction of Witten in [14] lifts a configuration of fourbranes stretched between fivebranes in Type IIA string theory to M-theory. The important feature here is that the fourbranes themselves are secretly fivebranes of M-theory wrapped around the eleventh dimension. The configuration of intersecting branes in ten dimensions is a projection of an eleven-dimensional configuration with a single fivebrane wrapped around a two-dimensional Riemann surface. The effective worldvolume theory on the fivebrane becomes four dimensional and is precisely the $\mathcal{N} = 2$ gauge theory whose

¹ For a geometrical engineering approach to $\mathcal{N} = 1$ theories see [18].

² The treatment of exceptional gauge groups has posed more problems [28].

low-energy effective action is described by that same Riemann surface. In the context of gauge theories these Riemann surfaces have been derived by a good-guess ansatz which could be justified by performing some highly nontrivial, physical consistency tests. The power of the brane construction lies in the fact that it gives a physical and comparatively straightforward derivation from “first principles” of these Riemann surfaces! While Witten restricted himself to the case of unitary gauge groups and products thereof this paper is devoted to the generalization these results to the case of the classical gauge groups $SO(N)$ and $Sp(2N)$. The new ingredient that we need is an orientifold. In adding an orientifold we follow recent work in [8]. $\mathcal{N}=1$ dualities from this point of view have been discussed in [8,11]. The role played by the Riemann surfaces in describing the nonperturbative corrections to the classical brane configurations in situations with $\mathcal{N}=2$ has been pointed out also in [8]. Our aims differ in that we are solely interested in deriving the Riemann surfaces of the $\mathcal{N}=2$ theories from D-brane considerations. In the course of doing this, using information from Type IIA string theory, M-theory and $\mathcal{N}=2$ gauge theories, we will obtain a rather interesting picture of the interaction of NS-fivebranes with an orientifold fourplane.

We will review Witten’s construction [14] in section two. Section three deals with the addition of an orientifold fourplane. Although an orientifold is an intrinsically stringy object we will argue that many of its properties (at least the properties crucial to our construction) carry over to M-theory. We derive the Riemann surfaces for the $\mathcal{N}=2$ theories with gauge groups $SO(N)$ and $Sp(2N)$ with and without matter hypermultiplets transforming in the fundamental representation of the gauge group. Witten pointed out that there are two mechanisms (albeit connected by a phase transition) for adding hypermultiplets. One consists of adding fourbranes stretching from a NS-fivebrane off to infinity, the other by adding D-sixbranes. The latter will be the subject of section four. Section five deals with generalizations of the elliptic models of [14]. Many new theories with vanishing beta-function emerge.

2. Review of Witten’s construction

The basic configuration of intersecting branes is schematically depicted in fig. 1. The fivebranes³ extend in the directions x^0, x^1, \dots, x^5 , are located at $x^7 = x^8 = x^9 = 0$ and at

³ We follow in our nomenclature [14] and speak only of four- five- and sixbranes dropping the specification NS or D since in Type IIA the dimension of the brane determines if it is NS or D.

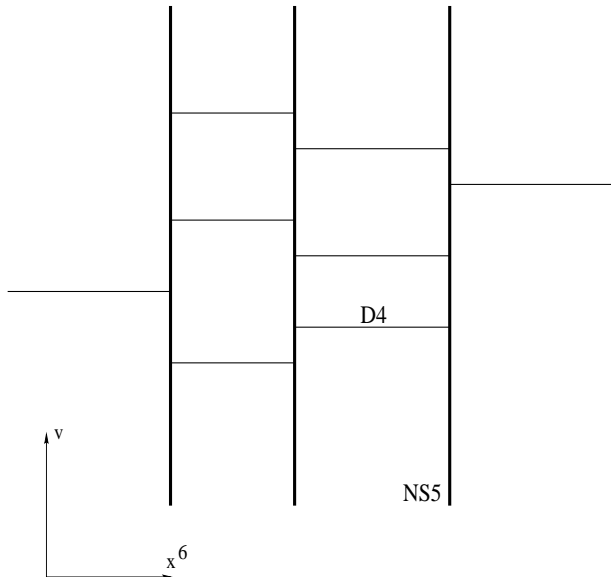


Fig. 1: A configuration of fivebranes connected by parallel fourbranes.

some arbitrary value of x^6 . The latter is only well defined in the classical approximation. We introduce the complex variable $v = x^4 + ix^5$. The fourbranes are stretched between the fivebranes. They extend over x^0, \dots, x^3 , live (classically) at a point in the v -plane and are finitely extended in x^6 . Since they are stretched between the fivebranes they live at the same point in the remaining dimensions. Sixbranes may also be present. These extend then in the directions $x^0, \dots, x^3, x^7, x^8, x^9$ and live at a point in x^4, x^5, x^6 .

The end of a fourbrane looks like a vortex on the fivebrane worldvolume. The position of the fivebrane in the x^6 direction becomes a scalar field in the worldvolume theory and obeys

$$\nabla^2 x^6 = \sum_i q_i \delta^2(v - a_i). \quad (2.1)$$

Here the a_i are the positions of the fourbranes ending on the fivebrane in consideration. The charges q_i are either $+1$ or -1 depending if the corresponding fourbrane extends to the left or to the right of the fivebrane in the x^6 direction. In section three we will argue that an orientifold fourplane parallel to the fourbranes induce charges ± 2 .

Because the fourbrane extends over three dimensions of the fivebrane, equation (2.1) is effectively a two-dimensional Poisson equation with solution

$$x^6 = k \sum_i q_i \log |v - a_i|. \quad (2.2)$$

These formulas are only valid in a kind of semiclassical approximation where one is at large $|v|$ and simultaneously far away from each fourbrane. A fivebrane has a well defined x^6 -value for $v \rightarrow \infty$ only if it is neutral, i.e. $\sum q_i = 0$. The constant k depends only on the Type IIA string coupling constant.

The contribution of the ends of the fourbrane to the energy of a fivebrane is given by a term in the action of the fivebrane of the form $\int d^4x d^2v \partial_\mu x^6 \partial^\mu x^6$, where the indices μ run over $0, \dots, 3$. For this to be finite it is necessary that

$$\sum_i a_i - \sum_j b_j = \text{const.}, \quad (2.3)$$

where a_i are the positions of the fourbranes to the left and b_j the positions of the fourbranes to the right.

This configuration is lifted to M-theory by taking into account the eleventh dimension, which we denote by x^{10} . It parameterizes a circle of radius R . Introducing the complex coordinate $s = (x^6 + ix^{10})/R$ equation (2.2) is generalized to

$$s = \sum_i q_i \log(v - a_i). \quad (2.4)$$

If there are k_α fourbranes in between the α -th and $(\alpha+1)$ -th fivebranes we will get a $SU(k_\alpha)$ gauge theory in the four dimensions x^0, \dots, x^3 . The overall $U(1)$ factor, usually present in gauge theories realized by D-branes, is frozen out. This is essentially due to the finite energy condition (2.3). Fourbranes to the left of the α -th fivebrane and to the right of the $(\alpha+1)$ -th fivebrane give rise to hypermultiplets in the fundamental of $SU(k_\alpha)$. In a configuration with $n+1$ fivebranes the gauge group is thus $\prod_{\alpha=1}^n SU(k_\alpha)$ with matter content transforming in the $(\mathbf{k}_1, \bar{\mathbf{k}}_2) \oplus (\mathbf{k}_2, \bar{\mathbf{k}}_3) \oplus \dots \oplus (\mathbf{k}_{n-1}, \bar{\mathbf{k}}_n)$ (we are assuming that there are no semi-infinite fourbranes at the ends of the chain of fivebranes). The constant in the finite energy condition (2.3) is a characteristic parameter of the α -th fivebrane giving rise to a bare mass to the hypermultiplets in the $(\mathbf{k}_\alpha, \bar{\mathbf{k}}_{\alpha+1})$. One can also compactify the x^6 direction by periodic identification of the $n+1$ -th fivebrane with the first one. Then there is an overall $U(1)$ -factor in the gauge group.

When we are going to place an orientifold fourplane we will always work on the covering space. For each fourbrane there is then a mirror image. This will have two related effects. The constant on the right hand side of (2.3) is forced to vanish thus there are no bare mass parameters in the theories with orientifolds. This restriction also arises from the gauge

theory point of view as we see later. Secondly, even upon compactifying the x^6 -direction we do not get an overall $U(1)$ factor in the gauge group. Although at first glance it might seem so, there is no loss of generality.

The gauge coupling constant of a $SU(k_\alpha)$ factor is essentially given by the distance between the α -th and $(\alpha+1)$ -th fivebranes. Taking into account the eleventh dimension of M-theory this gives rise to the formula

$$-i\tau_\alpha = s_\alpha - s_{\alpha-1}. \quad (2.5)$$

We have $\tau_\alpha = \frac{\theta_\alpha}{2\pi} + \frac{4\pi i}{g_\alpha^2}$. The difference in the positions of the fivebranes in x^{10} determines the theta angle of the α -factor of the gauge group. The scale of the gauge theory is set by v . At large v and using $i\tau = b_0 \log v$, we can read off the one loop beta function coefficient $b_{0,\alpha}$ for the $SU(k_\alpha)$ factor

$$b_{0,\alpha} = -2k_\alpha + k_{\alpha+1} + k_{\alpha-1}. \quad (2.6)$$

In M-theory the fivebranes and fourbranes are really the same object. What appears as fourbranes in Type IIA string theory are just M-theory fivebranes wrapped around the eleventh dimension. Thus the fourbranes are actually better thought of as tubes connecting the fivebranes. In this way one sees very directly that there is really only one fivebrane wrapped around a Riemann surface Σ . It is embedded in the two-dimensional complex space parameterized by v and s . There is a slight subtlety though: the Riemann surface obtained in this way extends to some points at infinity and is therefore noncompact. A compact one can be made by adding a finite number of points. It has already been shown in [2] that a fivebrane wrapped around a Riemann surface gives rise to a $\mathcal{N}=2$ $SU(k)$ gauge theory on the four-dimensional part of its world volume. Precisely the same mechanism is present here. BPS states are M-theory membranes of minimum area whose boundaries live on Σ .

The properties of low-energy supersymmetric effective field theory imply that the low-energy dynamics is described by an integrable system [29]. Examples of these integrable systems have been constructed in [29–32]. The integrable systems that appear in this context can be described as generalizations of the Hitchin system [33], which associates an integrable system to a complex curve Σ embedded in some complex two-dimensional symplectic manifold Q in the following way. Let Σ' be a curve in Q corresponding to a deformation of Σ and let \mathcal{L}' be a line bundle on Σ' . Then the deformation space of the pairs (Σ', \mathcal{L}') defines an integrable system [34]. They actually considered the case of

compact Σ , but the results can be extended to the noncompact case. The gauge theories we consider in this paper are all described by integrable systems of this type with Q identified appropriately either as flat space \mathbb{C}^2 , multi-Taub-NUT space, or the product $E \times \mathbb{C}$ (E an elliptic curve), and Σ identified with the curve describing the theory.

To compute the genus of the compactified Riemann surface we just have to count the number of tubes connecting the fivebranes, $g = \sum_{\alpha} (n_{\alpha} - 1)$. Here we are slightly more general than [14] in that we do not necessarily identify the number of fourbranes k_{α} with the number of tubes n_{α} connecting the fivebranes. What we have in mind is the following. On the fivebrane we have an antisymmetric tensor field with self-dual field strength T . Harmonic one forms Λ on Σ give rise to gauge fields in four dimensions through $T = F \wedge \Lambda + *F \wedge *\Lambda$. The gauge field is obtained by integrating T over one cycles of Σ . The number of harmonic one-forms on a Riemann surface equals its genus and their period integrals are points on the Jacobian of Σ . However, as emphasized in [30] the physics of the $\mathcal{N}=2$ gauge theory is not determined simply by the Jacobian of Σ but by a sub-abelian variety whose rank coincides with the rank of the gauge group. This is the so-called special Prym variety $\text{Prym}(\Sigma)$. Typically it is the subspace invariant under an involution of the Riemann surface. Therefore we have to take into account the possibility that there are more tubes connecting the fivebranes than arise simply from the fourbranes. Such tubes would be generated by nonperturbative effects. The gauge fields originating from them however are expected to play no physical role and should vanish upon projecting onto $\text{Prym}(\Sigma)$. In our cases this will be naturally achieved by orientifolding the configuration.

The physically relevant quantity of the low-energy gauge theory that the curves determine is the matrix of couplings $\tau^{\mu\nu}(u_l)$ as a function of the parameters of the Coulomb moduli space u_l , which appears as the period matrix of the curve. We take $\mu, \nu = 1, \dots, g$ where g is the genus of the Riemann surface Σ . If one also knows the Seiberg-Witten differential λ of the curve one may also derive the masses of all BPS states via the equation $M = |a_{\mu} q^{\mu} + a_D^{\mu} h_{\mu}|$, where q^{μ} are electric and h_{μ} are magnetic quantum numbers and a_D and a are given by integrals of λ , $a_D^{\mu} = \oint_{\alpha^{\mu}} \lambda$, and $a_{\nu} = \oint_{\beta_{\nu}} \lambda$. Here we use a basis of the $2g$ one-cycles $(\alpha^{\mu}, \beta_{\nu})$ on Σ with a standard intersection form $\langle \alpha^{\mu}, \beta_{\nu} \rangle = \delta_{\nu}^{\mu}$, $\langle \alpha^{\mu}, \alpha^{\nu} \rangle = \langle \beta_{\mu}, \beta_{\nu} \rangle = 0$.

Following [2] we may determine the Seiberg-Witten differential for all the cases considered in this paper. The BPS states in the fivebrane worldvolume theory arise as membranes whose boundary lies on the Riemann surface [15]. The BPS condition requires the worldvolume to be of the form $\mathbb{R} \times D$ where D is a complex Riemann surface of minimal area.

Let us consider here the case when the Riemann surface Σ is embedded in flat $Q = \mathbb{R}^3 \times S^1$. It is convenient to introduce the single valued variable $t = \exp(-s)$. The area of D is given by

$$V \sim \left| \int_D \frac{dv \, dt}{t} \right| = \left| \int_{\partial D} \frac{v \, dt}{t} \right|. \quad (2.7)$$

We should therefore set $\lambda = v \, dt/t$. This differential agrees with that obtained for $SU(N)$ gauge group with fundamental matter [19–22]. This form for the differential will also carry over to the all theories considered in the following section.

The precise form of the Riemann surface can be obtained by relatively simple considerations. It will be described by an equation $F(t, v) = 0$. For fixed v the zeros of $F(t, v)$ are the positions of the fivebranes. For fixed t the solutions in v indicate the presence of the fourbranes. $F(t, v)$ can be determined uniquely by using the information about the bending of the fivebranes. At large v the solutions in t should have the form $t_\alpha = c_\alpha v^{a_\alpha}$ where $-a_\alpha$ is the sum of the charges sitting on the α -th fivebrane, $-a_\alpha = \sum_i q_{i,\alpha}$ and c_α is some constant. Semi-infinite fourbranes sitting to the left or to the right of a fivebrane will bend this five brane to $x^6 = \pm\infty$. If v equals the position of such a semi-infinite fourbrane $F(t, v)$ must have $t = 0$ or $t = \infty$ as solution at this particular value of v . For a situation with only two fivebranes and k fourbranes in between them these considerations determine $F = t^2 + B(v)t + 1$. $B(v)$ is of the form $v^k + u_2 v^{k-2} + \dots u_k$ where a shift in v has been performed to absorb the term proportional to v^{k-1} . This is equivalent to the curves of [20,21]. In the following we will show how one can determine the curves for the orthogonal and symplectic gauge groups by taking properly into account the effects of the orientifold.

3. Including an Orientifold

3.1. The Orientifold

In this section we want to extend the previous construction to $\mathcal{N}=2$ four-dimensional gauge theories with orthogonal and symplectic gauge groups. Orthogonal and symplectic groups can be obtained from D-brane configurations by introducing an orientifold projection. This consists of a projection that combines the process of modding out a space-time

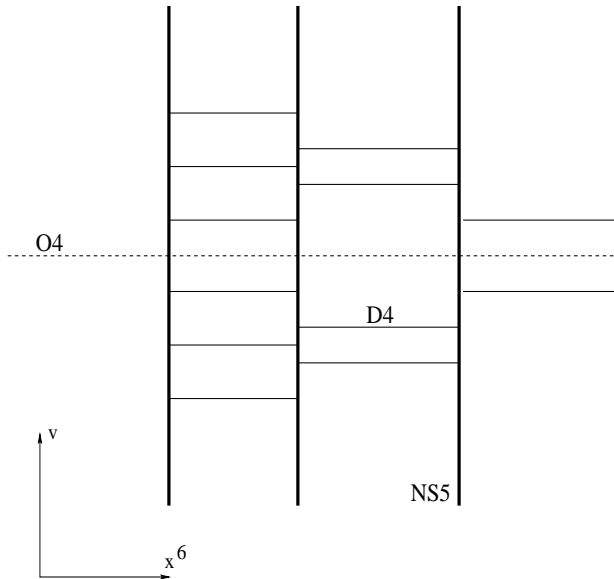


Fig. 2: A configuration of fourbranes and fivebranes and an orientifold fourplane.

symmetry and world-sheet parity inversion. The fixed points of the space-time symmetry define orientifold planes.

Orientifold projections generically break half of the existing supersymmetries. We can however introduce an orientifold plane in the D-brane configuration considered in the previous section without breaking any further supersymmetry. This can be achieved by placing an orientifold fourplane parallel to the fourbranes⁴. This corresponds to modding out by the space-time transformation:

$$(x^4, x^5, x^7, x^8, x^9) \rightarrow (-x^4, -x^5, -x^7, -x^8, -x^9). \quad (3.1)$$

Each object which does not lie at the fixed point of (3.1), over the orientifold plane, must have a mirror image. Since fourbranes joining mirror pairs of fivebranes would break supersymmetry, in our configurations all fivebranes will sit at $x^7 = x^8 = x^9 = 0$. Therefore as in the previous section, we can neglect the x^7, x^8, x^9 directions in describing the orientifolded brane configuration fig. 2.

The world-sheet parity projection Ω allows for $\Omega^2 = \pm 1$. The sign determines if we will obtain an orthogonal ($\Omega^2 = 1$) or symplectic ($\Omega^2 = -1$) gauge group [35]. Orientifold

⁴ It is also possible to place an orientifold sixplane parallel to the sixbranes [8,11]. In the following however we will not analyze that case, and will restrict to the presence of an orientifold fourplane.

planes behave as non-dynamical BPS objects, carrying a net charge under RR gauge fields. The orientifold charge⁵ depends on the type of parity projection, being ∓ 1 for $\Omega^2 = \pm 1$ in the normalization in which Dirichlet branes have charge 1.

As argued in [8], we will assume that the orientifold extends along the entire x^6 -direction and each time it crosses a fivebrane its charge changes sign. This can be justified by considering a configuration with $k > 2$ fivebranes in which an orientifold projection has been performed. Fivebranes in the presence of orientifolds have further been considered in [36]. Fourbranes ending to the left of the α -th fivebrane and to the right of the $\alpha+1$ -th fivebrane provide fundamental matter for the gauge theory on the fourbranes in between the α -th and $(\alpha+1)$ -th fivebranes. The flavor group for $SO(N_c)$ or $Sp(N_c)$ gauge theories is constrained to be respectively $Sp(2N_f)$ or $SO(2N_f)$ [27]. Since gauge and flavor groups are interchanged when we move from one set of fourbranes to the next, we see that the orientifold should change nature each time it crosses a fivebrane and should therefore extend along the whole configuration.

In the next subsections we will propose a way of deriving the exact solution for $\mathcal{N}=2$ four-dimensional gauge theories based on orthogonal and symplectic groups by lifting Type IIA brane configurations to M-theory along the lines of [14]. The first question that is raised is how to describe an orientifold plane in M-theory. Our orientifold fourplane is charged under the same RR gauge field as Dirichlet fourbranes. We will assume that, as Dirichlet fourbranes, it corresponds to a six-dimensional object in M-theory which is wrapped around the eleventh direction. The details of this picture will be explained further in the following.

3.2. $SO(2k)$ Gauge Groups

In this subsection we consider orthogonal groups with an even number of colors, $SO(2k)$. We will restrict to pure Yang-Mills theories. Our space-time is now an orbifold, we will however work on the covering space by considering \mathbb{Z}_2 invariant configurations.

The simplest configuration consists of 2 fivebranes traversed by the orientifold plane, together with $2k$ fourbranes ending on the fivebranes. In order to describe the embedding of this configuration in M-theory it is sufficient to consider the 2-complex dimensional space $Q = \{v = x^4 + ix^5, s = x^6 + ix^{10}\}$. The orientifold sits at $v = 0$, and each fourbrane at v has a mirror image at $-v$.

⁵ In a toroidal space T^5 (3.1) would generate 32 orientifold planes. By sending the radius of each circle to infinity only one orientifold plane remains at a finite distance. This is the situation we will consider in the following.

We expect that in M-theory, the set of fourbranes and fivebranes can be described in terms of a single M-fivebrane wrapped around a certain Riemann surface. Using the variable $t = e^{-s}$, the Riemann surface will be given in our case by $F(t, v^2) = 0$. Since we have 2 fivebranes, F must be a quadratic polynomial in t

$$A(v^2) t^2 + B(v^2) t + C(v^2) = 0. \quad (3.2)$$

The first condition this curve must satisfy is to reproduce the correct bending of each of the fivebranes at large values of v . This bending is determined by the RR charge of the objects that end or traverse the fivebrane. The essential ingredient for the determination of F is that the orientifold plane, though it traverses the fivebranes, provides a net charge contribution to them. This is due to the fact that the orientifold charge changes sign when crossing a fivebrane. In the case we are considering, the orientifold is seen by the left fivebrane as a $+2$ charge, and as a -2 charge ⁶ by the right fivebrane. Therefore for $v \rightarrow \infty$ (3.2) should reduce to

$$t_i \sim v^{a_i}, \quad a_1 = -a_2 = 2k - 2, \quad (3.3)$$

where $2k$ is the number of fourbranes ending on each fivebrane.

We analyze now the locus $v = 0$, where the orientifold sits. A semi-infinite fourbrane ending to the left of the first fivebrane would be represented by (3.2) as the solution $t = \infty$ at the value of v where fourbrane and fivebrane, in the perturbative picture, meet. In the M-theory picture, the whole configuration is described as a single M-fivebrane. We could then view the fourbrane as the deformation of the fivebrane induced by the presence of a non-zero charge in its world-volume. Following this reasoning, we can treat the interaction between fivebrane and orientifold in the same way. Since the orientifold is seen by the fivebrane as a net charge contribution, it should deform the fivebrane in the direction determined by the sign of that charge. In our case, this implies that at $v = 0$ the first fivebrane will deform to $t = \infty$ and the second to $t = 0$. Therefore the quadratic equation (3.2) must have as a solution $v = 0$, with $t = 0$ and $t = \infty$. The curve that meets this requirement, together with (3.3), is

$$v^2 t^2 + B(v^2) t + v^2 = 0, \quad (3.4)$$

⁶ This has also been discussed in [11].

with B the most general polynomial of order k in v^2

$$B(v^2) = v^{2k} + u_2 v^{2k-2} + \cdots + u_{2k}. \quad (3.5)$$

The normalizations in (3.4) and (3.5) are fixed by conveniently rescaling v and t . Multiplying (3.4) by v^2 and redefining $\tilde{t} = v^2 t + B/2$, the previous curve reads

$$\tilde{t}^2 = \frac{B(v^2)^2}{4} - v^4, \quad (3.6)$$

which is the standard form for the curve that solves $\mathcal{N}=2$ $SO(2k)$ gauge theory without matter [25].

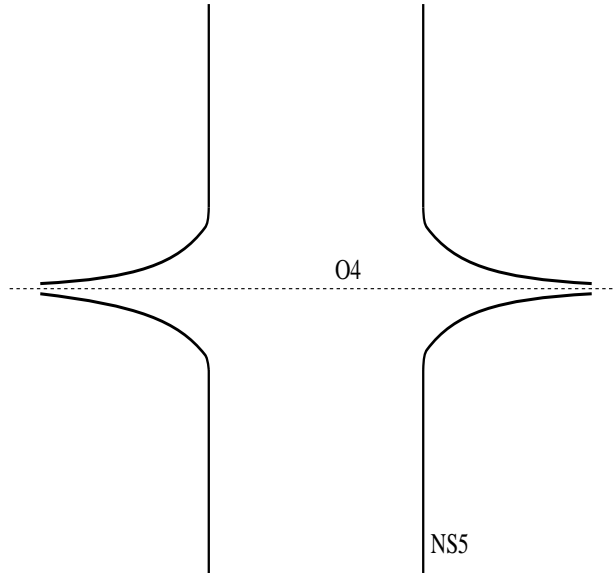


Fig. 3: Behavior of fivebranes near an orientifold plane that gives rise to $SO(2k)$ on fourbranes.

A somewhat strange feature of this family of curves is that there are singularities in its moduli space that do not correspond to massless BPS states. In fact the associated monodromy is trivial. This singularity sits at $u_{2k} = 0$ ⁷. In the brane configuration this corresponds to the situation when a mirror pair of fourbranes falls into the orientifold. The charges of this mirror pair are just enough to cancel the effective charge of the orientifold

⁷ There is a subtlety here in that the highest Casimir of $SO(2k)$ is actually reducible. More precisely, due to the structure of the Weyl group $u_{2k} = \tilde{u}^2$. The gauge invariant quantity \tilde{u} is the trace of the Pfaffian of the Higgs field in the adjoint in gauge theory language.

on the fivebrane so that the fivebranes are no longer bent. The effective gauge coupling squared is proportional to the inverse distance between the fivebranes. Thus we see that the gauge coupling at this particular singularity of the Riemann surface stays finite. This indicates that no additional BPS states become massless at this point⁸. Thus we get a nice physical picture why the singularity of the curve is not associated with massless states.

3.3. Symplectic Gauge Groups

We leave the study of $SO(2k+1)$ for the next subsection, and consider now symplectic gauge groups, $Sp(2k)$. The reason for this is that the treatment of $SO(2k)$ and $Sp(2k)$ groups share many common features.

In particular, we will consider the same configuration of branes that we used in the previous subsection. Namely, 2 fivebranes on which $2k$ fourbranes end. Each fourbrane at v will have a mirror image at $-v$ since we are working on the double cover of an orbifolded space. The sole difference will be that the orientifold plane sitting at $v = 0$ will have opposite charge assignments with respect to the $SO(2k)$ case.

Let us analyze first the bending of the fivebranes at large values of v . The orientifold contributes now with a charge -2 to the first fivebrane and a charge $+2$ to the second. Therefore we have

$$t_i \sim v^{a_i} \quad , \quad a_1 = -a_2 = 2k + 2. \quad (3.7)$$

We notice here a first difference. For unitary and orthogonal groups $a_2 - a_1 = b_0$, where b_0 was the one-loop beta function coefficient. However for symplectic groups we obtain

$$a_2 - a_1 = -(4k + 4) = 2b_0^{Sp}. \quad (3.8)$$

The extra factor 2 can be explained as a normalization effect intrinsic to the way in which orthogonal or symplectic groups are derived from orientifold constructions.

It is helpful now to have an explicit look on the orientifold construction of $SO(2k)$ and $Sp(2k)$ out of $SU(2k)$. Happily everything can be done in the semiclassical regime. The (classical) moduli space of an $SU(2k)$ gauge theory can be represented by $2k$ points moving in a complex plane. Gauge symmetry enhancement corresponds to colliding points

⁸ A finite coupling constant can also correspond to a singular point in moduli space with massless vectors and hypermultiplets whose contributions to the logarithmic divergence cancel each other. However the case we are analyzing can not provide this matter content, since the configuration we have does not include semi-infinite fourbranes or sixbranes.

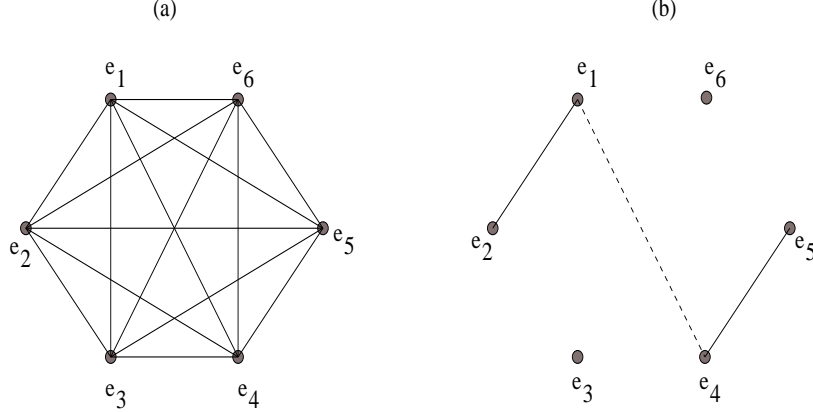


Fig. 4: (a) The vanishing cycles for the $SU(n)$ case. (b) After performing an orientifold projection only certain linear combinations survive. Only two of these cycles are shown. Solid lines correspond to cycles surviving both $SO(2k)$ and $Sp(2k)$ projections. The dashed line corresponds to a long root generator of $Sp(2k)$.

(vanishing 0-cycles). Of course we have a direct realization of this in terms of the endpoints of the fourbranes moving in the v -plane! A typical situation for $k = 3$ is shown in figure fig. 4 (a). A simple basis of vanishing cycles is given by $e_1 - e_2$, $e_2 - e_3$, $e_3 - e_4$, $e_4 - e_5$ and $e_5 - e_6$. Their intersection form coincides with the Cartan matrix of $SU(6)$. The orientifold projection selects now cycles that are odd under the space reflection⁹ $v \rightarrow -v$ fig. 4 (b). For the case of an $SO(2k)$ projection a simple basis of vanishing cycles is given by $e_1 - e_2 - e_4 + e_5$, $e_2 - e_3 - e_5 + e_6$ and $e_4 - e_1 - e_2 + e_5$. Their intersection form is proportional to the Cartan metric of $SO(6)$. For the case of an $Sp(2k)$ projection a simple basis is given by the first two of the former cycles and in addition $2(e_1 - e_4)$. The intersection form of these cycles is proportional to the metric on root space of $Sp(6)$. The important point is that there is a subset of cycles left invariant by both orientifold projections. However in the case of $SO(2k)$ these cycles give the long roots whereas the same cycles correspond to the short roots in the case of $Sp(2k)$. If we fix the normalization of the common roots to one, as is usual in physics, the length-squared of the long roots in the symplectic group will be two. The one loop beta function coefficient is given by a sum over the indices of the representations under which the various fields in a gauge theory transform. These indices depend however upon the normalization of the roots, e.g. for the adjoint $C_\theta = (\theta, \theta)g^\vee$

⁹ There is an additional minus for the vector excitation of the string such that precisely the vectors are left invariant.

where θ is the highest root and g^\vee the dual Coxeter number. This overall factor two can be absorbed into the definition of the gauge coupling of the $Sp(2k)$ theory.

Now we would like to derive the Riemann surface $F(t, v^2) = 0$ that provides the exact solution for $\mathcal{N}=2$ Yang-Mills theories with symplectic gauge groups. From (3.8), the curve should take the form

$$t^2 + (v^2 B(v^2) + c) t + 1 = 0. \quad (3.9)$$

To reproduce the correct bending of the fivebranes at infinity, $B(v^2)$ should be a polynomial even in v , of degree $2k$. A general polynomial of this form will depend on $k+1$ parameters. The coefficient of v^{2k} may be set to one by a rescaling of the gauge coupling. However, this leaves us with one parameter too many to describe the Coulomb branch of $Sp(2k)$ gauge theory. Let us analyze more carefully the behavior at $v = 0$. In the previous section we modeled the interaction between the fivebranes and the orientifold as the deformation induced on the fivebranes by the charge carried by the orientifold. This deformation depends on the sign (and the magnitude) of the charge. According to this, since now we are considering an orientifold with the opposite charges, the deformation of the fivebranes should point in the reverse direction.

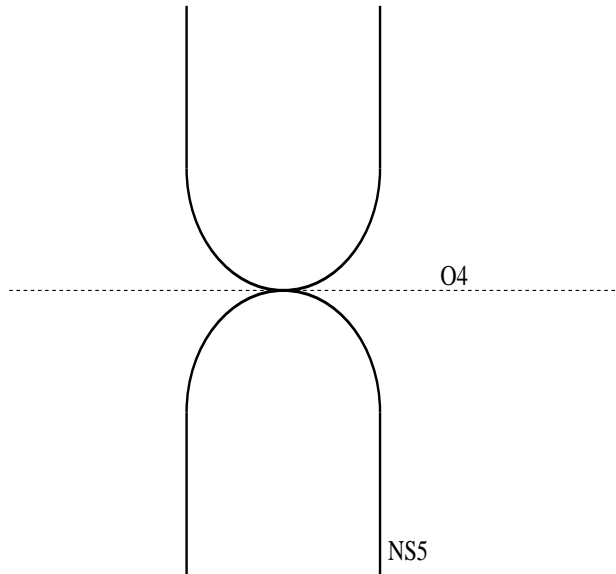


Fig. 5: Behavior of fivebranes near an orientifold plane that gives rise to $Sp(2k)$ gauge theory on fourbranes.

Namely, instead of running to $t = 0, \infty$ at $v = 0$ both fivebranes will be deformed towards each other, as in fig. 5. They will eventually meet at a central point and the charges will cancel. When this happens a single additional tube connects the two fivebranes. For this to happen in a way which preserves the $v \rightarrow -v, t \rightarrow t$ symmetry we must demand that (3.9) has a double root at $v = 0$. This condition implies the curve is of the form¹⁰

$$t^2 + (v^2 B(v^2) - 2) t + 1 = 0. \quad (3.10)$$

Shifting $\tilde{t} = t + (v^2 B(v^2) - 2)/2$, this Riemann surface can be rewritten as

$$\tilde{t}^2 = \frac{(v^2 B(v^2) - 2)^2}{4} - 1, \quad (3.11)$$

which is a double cover of the known curve solving $\mathcal{N}=2$ pure gauge theory with symplectic gauge group [27].

The degree of the polynomial is $4k + 4$ which indicates $2k + 2$ branchcuts. Due to the presence of the double point we are however in a degenerate configuration where two branch-points coincide at $v = 0$ and two branchcuts melt into a single one. Therefore the genus of the curve is $2k$ instead of $2k + 1$. In the usual representation of hyperelliptic curves the tubes connecting the fivebranes are represented by branchcuts. Here we have one more branchcut than expected from the number of fourbranes in the semiclassical analysis. This means that nonperturbative effects have generated the additional tube between the fivebranes. If we compute the discriminant of our curve for $Sp(4)$ with $u_2 = u$ and $u_4 = w$ we find

$$\Delta_{Sp(4)} = \Lambda^{12} w (u^2 + 4w) (-27\Lambda^{12} - 4\Lambda^6 u^3 - 18\Lambda^6 u w + u^2 w^2 + 4w^3). \quad (3.12)$$

Comparing this to what one gets from the curve in [27] one notes that there is an additional overall factor w present in our case. This additional singularity arises when the branchcut connecting the mirror images of branchcuts in fig. 6, shrinks to zero size. According to our argument in section two this should not correspond to a physical singularity. Indeed the cycle around this branchcut belongs to the even part of the homology

¹⁰ There is a misprint in the corresponding expression in [30]. It should read $(z - \mu/z)^2 + x^{(2l+2)} + x^{(2l)} u_2 + \dots + x^2 u_{2l}$. Our form of the curve agrees with this upon setting $t = z^2$ and $\mu = 1$. We thank N. Warner for a conversation on this point.

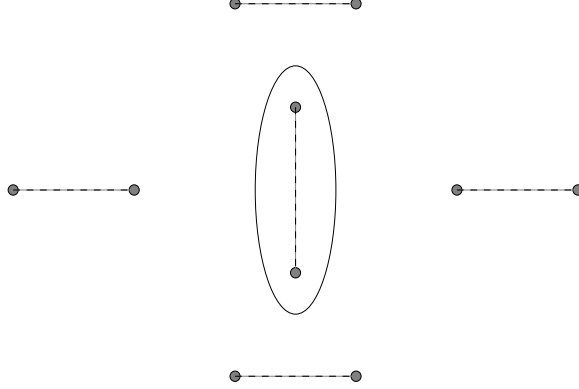


Fig. 6: Branchcuts for the curve describing $Sp(4)$ Yang-Mills.

under $v \rightarrow -v$. The orientifold projection however selects the odd part. We find therefore another type of apparent singularity in the curves for the symplectic gauge groups!

Let us stress the consistency of the picture we have obtained. The fivebranes are pushed towards each other by the orientifold. Once they meet, the configuration is stabilized since the orientifold is seen by each fivebrane as a charge of equal magnitude but opposite sign, i.e. ± 2 . Thus the region where the orientifold would have charge 1 shrinks to zero by nonperturbative effects. We are left with an orientifold plane which does not change nature even though it traverses the fivebranes. The same situation was encountered in the previous subsection. The orientifold plane in that case pushed the fivebranes off each other at $v = 0$. With nothing to stabilize them they run out to $s = \pm\infty$, extending the region where the orientifold has charge -1 all along the s -direction.

3.4. $SO(2k+1)$ Gauge Groups

In order obtain $SO(2k+1)$ gauge groups we will consider the same brane configuration as in subsection (3.1) but with an additional fourbrane lying over the orientifold. This new brane will be taken to end on the fivebranes, in the same way as the $2k$ paired fourbranes. The new single fourbrane is frozen at $v = 0$ since it does not have a mirror image.

In this case, the bending of the fivebranes at large v is given by

$$t_i \sim v^{a_i}, \quad a_1 = -a_2 = 2k - 1, \quad (3.13)$$

because the orientifold and the additional fourbrane cancel charge in the interval between fivebranes. Now the system of orientifold plus fourbrane at $v = 0$ is seen by the fivebranes as a charge $+1$ on the left fivebrane and -1 on the right. Still these charges will deform the

fivebranes off each other to $t = 0, \infty$, as happened for the $SO(2k)$ groups. The Riemann surface that reproduces the expected behaviors at $v \rightarrow \infty$ and $v = 0$ is

$$v t^2 + B(v^2) t + v = 0, \quad (3.14)$$

with $B(v^2)$ as in (3.5). Multiplying (3.14) by v and redefining $\tilde{t} = vt + B/2$, we get

$$\tilde{t}^2 = \frac{B(v^2)}{4} - v^2 \quad (3.15)$$

which agrees with the spectral curve for $SO(2k+1)$ Yang-Mills theories [23].

While the Riemann surface in the form (3.15) is invariant under $v \rightarrow -v$, (3.14) is only invariant under the combined operation $v \rightarrow -v, t \rightarrow -t$. In terms of the (v, s) variables, this corresponds to modding out by the transformation

$$\begin{aligned} v &\rightarrow -v, \\ s &\rightarrow s + i\pi \end{aligned} \quad (3.16)$$

combined with worldsheet parity reversal.

3.5. Effects of Semi-Infinite Fourbranes

We can now add mirror pairs of semi-infinite fourbranes to the left or to the right of the previous configurations. Whenever v equals the position of such a fourbrane $F(t, v)$ should have $t = \infty$ or $t = 0$ as a solution. Again this determines the curves uniquely. For the orthogonal groups one finds

$$\begin{aligned} F_{SO(2k)} &= v^2 t^2 \prod_{i=1}^{N_f^L} (v^2 - m_i^2) + eB(v^2)t + f v^2 \prod_{j=1}^{N_f^R} (v^2 - m_j^2) = 0, \\ F_{SO(2k+1)} &= v t^2 \prod_{i=1}^{N_f^L} (v^2 - m_i^2) + eB(v^2)t + f v \prod_{j=1}^{N_f^R} (v^2 - m_j^2) = 0. \end{aligned} \quad (3.17)$$

In the case of symplectic gauge groups we have one more condition. The fivebranes should meet at $v = 0$. We therefore take as an ansatz

$$F_{Sp(2k)} = \prod_{i=1}^{N_f^L} (v^2 - m_i^2) t^2 + e(v^2 B(v^2) - c)t + f \prod_{j=1}^{N_f^R} (v^2 - m_j^2) = 0. \quad (3.18)$$

Here we have allowed for arbitrary constants e and f . As long as the β -function of the theories is negative we can set these constants to one by rescaling t and v . Once this is done, the constant c is fixed by demanding that F have a double root at $v = 0$ and turns out to yield $c = 2 \prod_{i=1}^{N_f^L} \prod_{j=1}^{N_f^R} m_i m_j$. In all cases it is easy to find a transformation that brings these curves into the already known forms as summarized in [37],

$$\text{SO}(2k) : \quad v^2 t \prod_{i=1}^{N_f^L} (v^2 - m_i^2) = \tilde{t} - \frac{B(v^2)}{2} \quad (3.19a)$$

$$\text{SO}(2k+1) : \quad vt \prod_{i=1}^{N_f^L} (v^2 - m_i^2) = \tilde{t} - \frac{B(v^2)}{2} \quad (3.19b)$$

$$\text{Sp}(2k) : \quad t \prod_{i=1}^{N_f^L} (v^2 - m_i^2) = \tilde{t} - \frac{v^2 B(v^2) - c}{2} \quad (3.19c)$$

Let us now ask what happens if we place a single semi-infinite fourbrane on top of the orientifold plane to the right of a configuration with symplectic gauge group. Such a configuration gives only half as many states as are needed to form a hypermultiplet in the fundamental of the Sp gauge group. These states will form a half hypermultiplet. A mass term is not possible for half hypermultiplets. In our brane configuration such a term would correspond to moving the semi-infinite fourbrane off the orientifold. Naively applying the same arguments as in the previous cases we find that the curve describing this configuration is

$$F_{1/2\text{hyper}} = t^2 + v^2 B(v^2) t + v. \quad (3.20)$$

However, this is not consistent since the curve is not symmetric under a \mathbb{Z}_2 symmetry taking $v \rightarrow -v$. The only way to achieve this now is to take $|v|$ instead of v in the last term of (3.20). Clearly this destroys the complex structure and therefore breaks supersymmetry. It is indeed well known that $\mathcal{N}=2$ theories with symplectic gauge groups and an odd number of half hypermultiplets suffer from Witten's global anomaly [38]. We interpret the inconsistencies of the curve for the brane configurations corresponding to these gauge theories as a manifestation of this global anomaly. That we can actually detect this through D-brane considerations shows how powerful this approach is.

In the case of vanishing β -function we have to take into account that we can absorb only one of the two constants e and f by scalings. The remaining parameter can be adjusted arbitrarily and determines the UV-value of the gauge coupling. Translated into

our brane configuration this states that the positions of the fivebranes are well-defined at large v . More precisely, one finds $t \sim \lambda_{\pm} v^a$ where $a = 2k$ or $a = 2k + 2$ for orthogonal and symplectic gauge groups respectively and λ_{\pm} are the roots of $y^2 + ey + f$. This is the same structure as in [14] and following the arguments of that paper the duality group is $\Gamma_0(2)$ in our cases as well.

The cases of positive β -function can be described as follows. For the gauge groups of the form $SO(2k)$ after using the transformation (3.19a) the curves are of the form

$$\tilde{t}^2 = e^2 B^2(v^2) - f v^4 \prod_{i=1}^{N_f} (v^2 - m_i^2). \quad (3.21)$$

For $N_f > 2k - 2$ the asymptotic behavior of the fivebranes shows that they become parallel at large v . It is not changed if we add terms in $B(v^2)$ ¹¹. The highest term we can add in this manner is of order $2k' = N_f + 2$ if N_f is even and of order $2k' = N_f + 1$ if N_f is odd. If N_f is even the resulting curve is the one of the $SO(N_f + 2)$ theory with vanishing β -function. The physical interpretation is clear: the UV strongly coupled theory is embedded in the theory with smallest gauge group of the same family with finite UV-behavior. If N_f is odd there is no way the theory can be deformed into a usual gauge theory without changing the asymptotic behavior. The gauge group within the same family can be at most enlarged to $SO(N_f + 1)$. In fact the positions of the fivebranes go as $t_{\pm} = \pm \sqrt{-f} v^{N_f+2}$. The gauge coupling is proportional to $\log(t_+/t_-)$. The theory seems to flow to a genuinely strongly coupled fixed point with no adjustable free parameter. This is essentially the same behavior as found in [14] for unitary gauge groups.

For symplectic gauge groups the story is similar. Here we need $N_f > 2k + 2$. Again, if N_f is even this theory flows to the one with vanishing β -function and N_f flavors. For N_f odd there is a similar strongly coupled fixed point. For gauge groups of the form $SO(2k + 1)$ the behavior is just the opposite. If N_f is odd one can embed the theory into a UV-finite one with gauge group $SO(N_f + 2)$. The strongly coupled fixed point appears for N_f even.

¹¹ Here we do not admit terms that would break the symmetry $v \rightarrow -v$.

3.6. Product Groups

In this subsection we want to derive the exact solution for gauge theories whose gauge group is a product of orthogonal and symplectic groups. We start with the case where the gauge group involves $SO(2k)$ and $Sp(2k)$ groups only.

Gauge Groups of the form $\cdots \times SO(2k_{\alpha-1}) \times Sp(2k_{\alpha}) \times SO(2k_{\alpha+1}) \times \cdots$

In order to get product groups we need to consider configurations with more than two fivebranes. We will consider a chain of $n + 1$ fivebranes, with $2k_{\alpha}$ and $2k_{\alpha+1}$ fourbranes ending respectively on the left and right of the α -th fivebrane ($\alpha = 0, \dots, n$). We will assume that there are no semi-infinite fourbranes on the ends of the configuration, therefore $k_0 = k_{n+1} = 0$. As before an orientifold plane will traverse the whole configuration at $v = 0$. In the Type IIA string picture each time the orientifold crosses a fivebrane it changes nature. The product group structure we obtain is

$$\cdots \times SO(2k_{\alpha-1}) \times Sp(2k_{\alpha}) \times SO(2k_{\alpha+1}) \times \cdots. \quad (3.22)$$

We also get $n - 1$ half hypermultiplets transforming as $(\mathbf{2k}_{\alpha}, \mathbf{2k}_{\alpha+1})$, where $\mathbf{2k}_{\alpha}$ denotes the fundamental representation of the corresponding orthogonal or symplectic gauge group, G_{α} . With respect to each G_{α} there is always an even number of half hypermultiplets present.

In the following we want to consider theories with negative or zero β functions. As in the previous sections, we can read the one-loop β function coefficient from the bending of the fivebranes at large v

$$t_{\alpha} \sim v^{a_{\alpha}} \quad , \quad a_{\alpha} = 2k_{\alpha+1} - 2k_{\alpha} - 2\omega_{\alpha}, \quad (3.23)$$

where ω_{α} is the charge of the orientifold on the left of the α -th fivebrane. From (2.5), the one-loop β -function coefficient $b_{0,\alpha}$ for the gauge group G_{α} is proportional to $a_{\alpha} - a_{\alpha-1}$. Thus the condition $b_{0,\alpha} \leq 0$ implies

$$a_0 \geq a_1 \geq \cdots \geq a_n. \quad (3.24)$$

The exact solution of this model will be given in terms of a Riemann surface $F(t, v^2) = 0$, where now F is a polynomial of order $n + 1$ in t

$$P_0(v^2) t^{n+1} + P_1(v^2) t^n + \cdots + P_{n+1}(v^2) = 0. \quad (3.25)$$

The relation (3.24) allows one to determine the degree of the polynomials $P_i(v^2)$. This can be seen by rewriting (3.25) as

$$P_0(v^2) \prod_{\alpha=0}^n (t - t_\alpha(v^2)) = 0, \quad (3.26)$$

where t_α are now rational functions of v^2 with the asymptotic behavior (3.23). Therefore the degree of P_i is

$$p_i = \sum_{j=0}^{i-1} a_\alpha + p_0 = 2k_i - (1 - (-1)^i) \omega_0 + p_0, \quad (3.27)$$

with p_0 the degree of P_0 . Since we assumed that there are no semi-infinite fourbranes on either end of the configuration, P_0 depends only on how the orientifold deforms the leftmost fivebrane. If $\omega_0 = 1$ the first factor in the chain is an orthogonal group, in which case the orientifold will push the first fivebrane to $t = \infty$. Using the results of subsection (3.2) we have then $p_0 = 2$. If $\omega_0 = -1$ the first group of the chain is symplectic. The orientifold will deform the first fivebrane towards the next one. In this case t remains finite at $v = 0$ and therefore $p_0 = 0$. We can now write (3.27) in the simple form

$$\begin{aligned} p_i &= 2k_i + 1 + (-1)^i, & \omega_0 &= 1, \\ p_i &= 2k_i + 1 + (-1)^{i+1}, & \omega_0 &= -1. \end{aligned} \quad (3.28)$$

It is convenient to explicitly mention the four different cases we can obtain. If there is an even number of fivebranes the first and last groups of the chain will belong to the same series. More precisely, for $\omega_0 = 1$ we get the chain $SO(2k_1) \times Sp(2k_2) \times \cdots \times SO(2k_n)$. For $\omega_0 = -1$ we will obtain $Sp(2k_1) \times SO(2k_2) \times \cdots \times Sp(2k_n)$. If there is an odd number of fivebranes the first and last groups will differ. When $\omega_0 = 1$ we derive the chain $SO(2k_1) \times Sp(2k_2) \times \cdots \times Sp(2k_n)$. On the contrary, when $\omega_0 = -1$ we will find $Sp(2k_1) \times SO(2k_2) \times \cdots \times SO(2k_n)$. In order to show the structure of the solution we will consider this last case, i.e. $\omega_0 = -1$ and $n + 1$ odd. Knowing this case, the others can be obtained in a straightforward way.

Using (3.28), the Riemann surface (3.25) for $\omega_0 = -1$ and $n + 1$ odd looks like

$$t^{n+1} + (v^2 B_1(v^2) + c_1) t^n + B_2(v^2) t^{n-1} + \cdots + v^2 + c_{n+1} = 0. \quad (3.29)$$

The functions B_i are generic polynomials even in v of degree $2k_{\alpha_i}$. The coefficients of the highest order term of B_i determine the asymptotic behavior of the fivebranes and should be

interpreted as coupling constants [14]. One of them can be eliminated by rescaling v . The remaining coefficients in the B_i can be interpreted as the Casimirs of G_{α_i} . c_1, c_3, \dots, c_{n+1} are constants that can be determined from the expected behavior at $v = 0$. In the case we are now considering, the orientifold pushes the rightmost fivebrane off to infinity. Therefore $t = 0$ must be a solution of (3.29) at $v = 0$, which fixes $c_{n+1} = 0$. The analysis of subsection (3.3) tells us that, with the exception of $t = 0$, all the other roots of (3.29) at $v = 0$ should be double roots

$$t^n + c_1 t^{n-1} + u_{2k_2}^{(2)} t^{n-2} + \dots + u_{2k_n}^{(n)} = \prod_{i=1}^{n/2} (t - t_i)^2, \quad (3.30)$$

where u_{2k_α} is the square of the exceptional Casimir of order k_α of $G_\alpha = SO(2k_\alpha)$. Condition (3.30) completely determines the constants c_i in terms of u_{2k_α} .

Contrary to the case of the $SU(k)$ product groups considered in [14], in our case we can not introduce bare mass parameters for the $(\mathbf{2k}_\alpha, \mathbf{2k}_{\alpha+1})$ half-hypermultiplets. Bare mass parameters correspond to a non-zero constant in (2.3) which is not allowed by the \mathbb{Z}_2 symmetry $v \rightarrow -v$ of our configurations. However this fact does not represent a lack of generality of the brane construction, since it can equally be derived from pure gauge theory considerations.

It is enough to analyze the case $G = Sp(2k_1) \times SO(2k_2)$ and one half-hypermultiplet $(\mathbf{2k}_1, \mathbf{2k}_2)$. Let us use $\mathcal{N} = 1$ superspace notation and represent the half-hypermultiplet by X_a^i , with $a = 1, \dots, 2k_1$ and $i = 1, \dots, 2k_2$. In this example the flavor group has been completely gauged. The only way to write a gauge-invariant mass term is

$$m X_a^i X_b^j \delta_{ij} J^{ab}, \quad (3.31)$$

where J^{ab} and δ_{ij} are the invariant matrices associated to symplectic and orthogonal groups respectively. Since $J^{ab} = -J^{ba}$ is antisymmetric while δ_{ij} is symmetric, the bare mass term (3.31) is identically zero. However the $4k_1 k_2$ hypermultiplets X_a^i can acquire masses by turning on Higgs expectation values. This corresponds to the $\mathcal{N} = 1$ superpotential

$$W = X_a^i X_b^j \phi_{Sp}^{ab} \delta_{ij} + X_a^i X_b^j \phi_{ij}^{SO} J^{ab}, \quad (3.32)$$

where ϕ_{Sp}^{ab} and ϕ_{ij}^{SO} are chiral $\mathcal{N} = 1$ fields in the adjoint representation of $Sp(2k_1)$ and $SO(2k_2)$ respectively. In all the cases we treat in this subsection a maximal subgroup of the flavor group has been gauged, therefore the same field theory argument implies that bare mass parameters are not allowed.

Let us analyze in more detail the curve for $G = Sp(2k_1) \times SO(2k_2)$. The Riemann surface (3.29) for this case is

$$t^3 + (ev^2 B_{Sp}(v^2) - 2 \prod_{i=1}^{k_2} a_{SO,i}) t^2 + B_{SO}(v^2) t + v^2 = 0. \quad (3.33)$$

We choose to eliminate the coefficient of $v^{2k_2}t$. Both polynomials B_{Sp} and B_{SO} can be written as

$$B_G(v^2) = \prod_{i=1}^{k_i} (v^2 - a_{G,i}^2), \quad (3.34)$$

where $\pm a_{G,i}$ represent the (classical) positions of the fourbranes in the v -plane. The curve (3.33) is gauge invariant only because for orthogonal groups of the form $SO(2k)$ the product $\prod_{i=1}^k a_i$ is a gauge invariant quantity!

We want now to take the leftmost or the rightmost fivebrane to $\pm\infty$ and recover from (3.33) the curves solving even orthogonal and symplectic gauge groups with matter in the fundamental representation. This will provide a check of the curves we have proposed. If we send the third fivebrane to infinity, the $SO(2k_2)$ gets effectively frozen since its classical gauge coupling is sent to zero by this process. The $2k_2$ fourbranes with boundary on the second and third fivebranes become now semi-infinite fourbranes. In this way we reduce to $Sp(2k_1)$ gauge group, plus k_2 $\mathcal{N}=2$ hypermultiplets in the fundamental representation. In (3.33) we have set the scales to one, alternatively we could have considered

$$t^3 + (ev^2 B_{Sp}(v^2) - 2 \prod_{i=1}^{k_2} a_{SO,i}) t^2 + B_{SO}(v^2) t + \Lambda v^2 = 0. \quad (3.35)$$

By sending $\Lambda \rightarrow 0$ we effectively take the third fivebrane to infinity, i.e. $t = 0$. In this limit $\prod_{i=1}^{k_2} a_{SO,i}$ behaves as the product of the bare masses of the k_2 fundamental hypermultiplets. Using (3.34), the curve (3.35) becomes

$$t^2 + (ev^2 B_{Sp}(v^2) - 2 \prod_{i=1}^{k_2} a_{SO,i}) t + \prod_{i=1}^{k_2} (v^2 - a_{SO,i}^2) = 0. \quad (3.36)$$

The factor e can now be set to one by scaling v and t appropriately and the expression coincides with the curves for symplectic gauge groups.

In the same way, if we send the first fivebrane to $-\infty$, we will reduce the gauge group to $SO(2k_2)$, plus k_1 hypermultiplets in the fundamental representation. As before we can introduce a scale Λ in (3.33) which corresponds to shifting the position of the first fivebrane.

However we must be careful to do it in a way that preserves the condition (3.30). This can be achieved by

$$\Lambda^2 t^3 + (ev^2 B_{Sp}(v^2) - 2\Lambda u_{2k_2}) t^2 + B_{SO}(v^2) t + v^2 = 0. \quad (3.37)$$

The limit $\Lambda \rightarrow 0$ corresponds to sending the first fivebrane to $s = -\infty$, i.e. $t = \infty$. Using again (3.34) and scaling v and t , (3.37) allows us to recover the solution for the orthogonal groups

$$\prod_{i=1}^{k_1} (v^2 - a_i^2) t^2 + B_{SO}(v^2) t + v^2 = 0. \quad (3.38)$$

The solutions for gauge groups of the form $SO(2k_1) \times Sp(2k_2) \times \cdots SO(2k_n)$ can be worked out in a similar way and take the form

$$F(t, v^2) = v^2 t^{n+1} + B_1(v^2) t^n + (v^2 B_2(v^2) + c_1) t^{n-1} + \cdots + B_n(v^2) t + v^2. \quad (3.39)$$

Here $n+1$ is an even integer. At $v=0$ this has $t=0$ and $t=\infty$ as solutions. Again we demand the other zeroes at $v=0$ to be double points. This fixes the $(n-1)/2$ constants c_i .

For gauge groups of the form $Sp(2k_1) \times SO(2k_2) \times \cdots Sp(2k_n)$ one finds

$$F(t, v^2) = t^{n+1} + (v^2 B_1(v^2) + c_1) t^n + B(v^2) t^{n-1} + \cdots + (v^2 B(v^2) + c_n) t + 1. \quad (3.40)$$

Again $n+1$ is an even integer and the $(n+1)/2$ constants c_i are fixed by demanding that we double points only at $v=0$.

Gauge Groups of the form $\cdots \times SO(2k_{\alpha-1} + 1) \times Sp(2k_{\alpha}) \times SO(2k_{\alpha+1} + 1) \times \cdots$

It is also possible to have gauge groups consisting of factors of the form $SO(2k_1 + 1) \times Sp(2k_2) \times SO(2k_3 + 1)$. In this case one has two massless half hypermultiplets in the Sp groups and this gives a consistent theory. However, we still cannot give these half hypermultiplets a mass. We can understand this by taking the view of gauging the global flavor symmetry of a $\mathcal{N}=2$ gauge theory with symplectic gauge group. The flavor symmetry is always $SO(2N_f)$ with N_f being the number of hypermultiplets. The brane configuration corresponds to the case when one gauges only a $SO(2k_1 + 1) \times SO(2k_3 + 1)$ subgroup of $SO(2N_f)$, where of course $(k_1 + k_3 + 1) = N_f$. Giving masses to the hypermultiplets corresponds now to turning on vev's of the Higgs fields Φ_{SO} in the adjoint of $SO(2k_1 + 1)$ and $SO(2k_3 + 1)$ respectively. Since Φ_{SO} has to lie in the Cartan subalgebra

it is of the form $\Phi = \text{diag}(a_1 i\sigma_2, \dots, a_{k_i} i\sigma_2, 0)$ where we used the usual form of the Pauli matrix σ_2 . In $\mathcal{N} = 1$ superspace the superpotential of such a theory can be written as

$$W = X_a^i X_b^j \Phi_{Sp}^{ab} \delta_{ij} + X_a^i X_b^j J^{ab} \Phi_{ij}^{SO_1} + Y_a^i Y_b^j \Phi_{Sp}^{ab} \delta_{ij} + Y_a^i Y_b^j J^{ab} \Phi_{ij}^{SO_2}. \quad (3.41)$$

The fields transforming in the fundamental of the first and second SO factor are denoted with X and Y respectively, J^{ab} is the symplectic metric. In this way one sees directly that the two half hypermultiplets remain massless.

It is easy to generalize the discussion of the previous case to this situation. Without going into the details we state the resulting form of the curves for the gauge group being $SO(2k_1 + 1) \times Sp(2k_2) \times SO(2k_3 + 1) \times \dots \times SO(2k_n + 1)$

$$F(t, v) = vt^{n+1} + B_1(v^2)t^n + vB_2(v^2)t^{n-1} + B_3(v^2)t^{n-2} + \dots + v = 0, \quad (3.42)$$

where $n+1$ is an even integer. The curve respects the symmetry $v \rightarrow -v$, $t \rightarrow -t$. We find that the number of relevant parameters in (3.42) matches precisely the number of Casimirs and couplings of the gauge group factors.

When deriving the curves associated with symplectic groups and product groups containing symplectic and even orthogonal groups we had to use a crucial ingredient. We asked that, except for possible solutions $t = 0$ or $t = \infty$, the roots of $F(t, 0) = 0$ should be double roots. This arose from requiring that when the fivebranes are deformed towards each other, they meet to form a single tube. Although for products of odd orthogonal and symplectic groups we do not require additional restrictions to fix the unique form of the curves (3.42), it should be pointed out the same picture holds in these cases as well. At $v = 0$ (3.42) has as solutions $t = 0, \infty$ and the roots of

$$u_{2k_1}^{(1)} t^{n-1} + u_{2k_3}^{(3)} t^{n-3} + \dots + u_{2k_n}^{(n)} = u_{2k_1}^{(1)} \prod_{i=1}^{(n-1)/2} (t^2 - t_i^2) = 0. \quad (3.43)$$

The solutions of (3.43) are of the form $t = \pm t_i$. However due to the \mathbb{Z}_2 symmetry $v \rightarrow -v$, $t \rightarrow -t$ these correspond to double points in the quotient space.

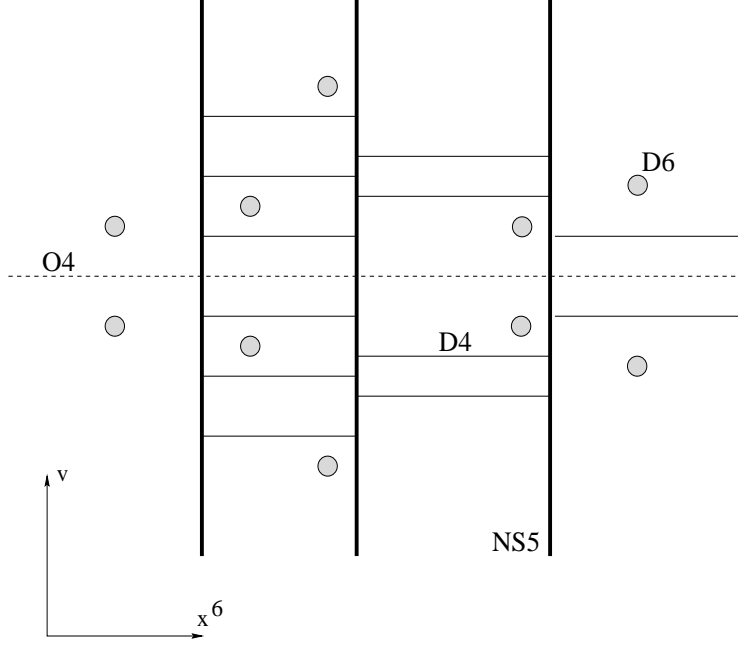


Fig. 7: A configuration of fourbranes, fivebranes and sixbranes.

4. Including Six-Branes

Now we consider the addition of sixbranes to the configurations of fourbranes and fivebranes previously discussed. The sixbranes are extended in the x^0, \dots, x^3 and x^7, x^8 and x^9 directions. Each sixbrane is accompanied by its image under the action of the orientifold symmetry.

An example of such a configuration is shown in fig. 7. Let d_α denote the number of pairs of sixbranes between the $(\alpha - 1)$ -th and α -th fivebrane. Open strings running between the sixbranes and fourbranes will give rise to d_α additional hypermultiplets in the fundamental representation of the gauge group.

A configuration of parallel sixbranes in M-theory corresponds to the product of a multi-Taub-NUT metric with flat \mathbb{R}^7 space [39]. The multi-Taub-NUT metric [40] takes the form

$$ds^2 = \frac{V}{4} d\vec{r}^2 + \frac{V^{-1}}{4} (d\tau + \vec{\omega} \cdot \vec{r})^2, \quad (4.1)$$

where

$$V = 1 + \sum_{a=1}^d \frac{1}{|\vec{r} - \vec{x}_a|}, \quad (4.2)$$

$$\vec{\nabla} \times \vec{\omega} = \vec{\nabla} V.$$

The \vec{x}_a are the positions of the sixbranes. It should be noted this eleven-dimensional solution is nonsingular. This will allow us to use the properties of low-energy M-theory to solve the low-energy physics of the gauge theory.

To construct the gauge theory curves we do not need the full details of the metric (4.1) but only a description of the space in one of its complex structures [14]. Such a description has been constructed in [41]

$$yz = P(v) = \prod_{a=1}^d (v^2 - e_a^2) . \quad (4.3)$$

This four-dimensional space replaces the flat $\mathbb{R}^3 \times S^1$ of the x^4, x^5, x^6, x^{10} directions of the previous sections.

The fourbranes behave as before under the action of the orientifold symmetry Ω . Depending on the choice $\Omega^2 = \pm$ we obtain orthogonal ($\Omega^2 = 1$) or symplectic gauge groups ($\Omega^2 = -1$). Ω^2 acts with the opposite sign on sixbranes, so the flavor symmetry group that arises is symplectic for ($\Omega^2 = 1$) and orthogonal for ($\Omega^2 = -1$). This restriction on the flavor symmetry groups, as we have seen before, is familiar from the field theory viewpoint.

The Seiberg-Witten differential for these gauge theories may be constructed following section 2. Now the Riemann surface is embedded in a curved spacetime. The area of the spatial component of the membrane worldvolume D will now be given by the formula

$$V \sim \left| \int_D \frac{dv \, dz}{\partial W / \partial y} \right| = \left| \int_{\partial D} \frac{v \, dz}{z} \right| , \quad (4.4)$$

where we have introduced $W(v, y, z) = zy - P(v)$. The Seiberg-Witten differential is then $\lambda = v \, dz/z$.

4.1. $SO(2k)$ with d Fundamentals

Let us first consider the case when we have a pair of fivebranes and d pairs of sixbranes. We wish to impose the condition that there are no semi-infinite fourbranes to the left or to the right of the fivebranes. The orientifold plane induces fourbrane charge on the fivebrane, as previously discussed. This means that as $v \rightarrow 0$, y must have a solution that goes as $1/v^2$ and one that goes as v^2 . The curve must also be invariant under the \mathbb{Z}_2 symmetry $v \rightarrow -v$, with y and z invariant. These conditions restrict the curve to the form

$$v^2 y^2 + B(v^2)y + v^2 C(v^2) = 0 . \quad (4.5)$$

Substituting $z = P(v^2)/y$, where $P(v^2) = \prod_{a=1}^d (v^2 - e_a^2)$ (remembering we have pairs of sixbranes positioned at $v = e_a$ and $v = -e_a$) we obtain

$$C(v^2)v^2z^2 + B(v^2)P(v^2)z + v^2P(v^2)^2 = 0 . \quad (4.6)$$

In order that z have only a solution that goes as $1/v^2$ and v^2 as $v \rightarrow 0$ we require that BP and P^2 are divisible by C . Taking the e_a 's to be distinct, the solution for C is

$$C = f \prod_{a=1}^{i_0} (v^2 - e_a^2)^2 \prod_{b=i_0+1}^{i_1} (v^2 - e_b^2) , \quad (4.7)$$

where i_0 and i_1 are integers and f is a constant. Following [14], i_0 will be the number of pairs of sixbranes to the left of the fivebranes, and $i_1 - i_0$ will be the number of pairs of sixbranes between the fivebranes. The solution for B takes the form

$$B = \tilde{B}(v^2) \prod_{a \leq i_0} (v^2 - e_a^2) , \quad (4.8)$$

for some polynomial \tilde{B} .

Defining $\tilde{y} = y / \prod_{a \leq i_0} (v^2 - e_a^2)$ the curve (4.5) becomes

$$v^2\tilde{y}^2 + \tilde{B}(v^2)\tilde{y} + f v^2 \prod_{a=i_0+1}^{i_1} (v^2 - e_a^2) = 0 . \quad (4.9)$$

When \tilde{B} is a polynomial of order k in v^2 , this is the curve describing $SO(2k)$ gauge group with d flavors of fundamental matter.

4.2. $Sp(2k)$ with d Fundamentals

The above arguments carry over in a straightforward way to the $Sp(2k)$ case. Now the orientifold projection gives rise to an opposite charge for the orientifold plane. y should therefore have no solutions that go to zero or infinity for finite v . Imposing the \mathbb{Z}_2 symmetry $v \rightarrow -v$, with y and z fixed, yields the curve

$$y^2 + B(v^2)y + C(v^2) = 0 . \quad (4.10)$$

Substituting in $z = P/y$, with P as defined above, we find

$$C(v^2)z^2 + B(v^2)P(v^2)z + P(v^2)^2 = 0 . \quad (4.11)$$

In order that z has no solutions that go to zero or infinity for finite v corresponding to semi-infinite fourbranes we must have that C divides BP and P^2 . The solution of these conditions is (4.7) and (4.8). Defining \tilde{y} and \tilde{B} as before we obtain the equation

$$\tilde{y}^2 + \tilde{B}\tilde{y} + f \prod_{a=i_0+1}^{i_1} (v^2 - e_a^2) = 0 . \quad (4.12)$$

As in the previous examples of $Sp(2k)$ gauge theories considered above, we must impose the condition that y has a double root at $v = 0$. This fixes

$$\tilde{B}(0) = 2f^{1/2} \prod_{a=i_0+1}^{i_1} ie_a . \quad (4.13)$$

$\tilde{B}(v^2)$ may then be written $v^2 B'(v^2) + \tilde{B}(0)$. Assuming B' is a polynomial in v^2 of order k , we find upon substituting back into (4.12) the familiar curve for $Sp(2k)$ with $i_1 - i_0$ fundamentals.

4.3. $SO(2k+1)$ with d Fundamentals

The only essential difference between this case and the $SO(2k)$ case is that now we have an additional fourbrane lying on the orientifold plane frozen at $v = 0$ and stretched between the fivebranes. This changes the asymptotic behavior of the fivebranes, so that now we must have a solution $y \sim 1/v$ and $y \sim v$ as $v \rightarrow 0$. The \mathbb{Z}_2 orientifold symmetry now acts as $v \rightarrow -v$, $y \rightarrow -y$, and $z \rightarrow -z$. These constraints imply the curve takes the form

$$vy^2 + B(v^2)y + vC(v^2) = 0 . \quad (4.14)$$

Following the same procedure as above, we can solve for B and C and we obtain the same answer (4.7) and (4.8). With \tilde{y} defined as before, the equation for the curve is

$$v\tilde{y}^2 + \tilde{B}\tilde{y} + v \prod_{a=i_0+1}^{i_1} (v^2 - e_a^2) = 0 . \quad (4.15)$$

Taking $\tilde{B}(v^2)$ to be a polynomial of degree k in v^2 , we obtain the curve for $SO(2k+1)$ with $i_1 - i_0$ fundamentals.

4.4. Product Gauge Groups

We now generalize to the models with $n + 1$ fivebranes. Depending on the orientifold projection, we can obtain alternating products of $Sp(2k)$ with $SO(2k')$ gauge groups, or $Sp(2k)$ with $SO(2k' + 1)$ gauge groups.

Product gauge groups of the form $Sp(2k_1) \times SO(2k_2) \times Sp(2k_3) \times \cdots \times SO(2k_n)$ with fundamental matter

The gauge group at the start of the chain is arranged to be $Sp(2k_1)$, while that at the end of the chain is chosen to be $SO(2k_n)$, implying that $n + 1$ is odd. Demanding that there be no semi-infinite fourbranes, means that there should be a solution $t \sim v^2$ as $v \rightarrow 0$, but no $t \rightarrow \infty$ solution in this limit. Further imposing symmetry under $v \rightarrow -v$ restricts the curve to the form

$$y^{n+1} + A_1(v^2)y^n + \cdots + v^2 A_{n+1}(v^2) = 0 . \quad (4.16)$$

Substituting $z = P/y$, we find

$$v^2 A_{n+1} z^{n+1} + A_n P z^n + \cdots + P^{n+1} = 0 . \quad (4.17)$$

The condition that there are no semi-infinite fourbranes leads to the conditions that $A_\alpha P^{n+1-\alpha}$ is divisible by A_{n+1} for all $0 \leq \alpha \leq n$. Following [14] we may then solve for the A_α

$$A_\alpha = g_\alpha(v^2) \prod_{s=1}^{\alpha-1} J_s^{\alpha-s} , \quad (4.18)$$

where g_α are polynomials and

$$J_s = \prod_{a=i_{s-1}+1}^{i_s} (v^2 - e_a^2) , \quad (4.19)$$

where the integers i_α are related to the number of sixbranes between the $(\alpha - 1)$ -th and α -th fivebrane by $d_\alpha = i_\alpha - i_{\alpha-1}$.

Finally we must impose the additional constraint that arises for Sp gauge groups in the case when we mod out by $v \rightarrow -v$ with y and z fixed, that the curve (4.16) have double roots in y at $v = 0$. This fixes $n/2$ of the constants appearing in the g_α for $\alpha = 1 \cdots n$, in terms of the other $n/2$. This gives us precisely the right number of parameters for the curve

(4.16) to describe the Coulomb branch of the $Sp(2k_1) \times SO(2k_2) \times Sp(2k_3) \times \cdots \times SO(2k_n)$ gauge group with $(\mathbf{2k}_\alpha, \mathbf{2k}_{\alpha+1})$ matter together with d_α fundamentals in the α factor.

The case when $n+1$ is even may be described in a very similar way. Now we have Sp gauge groups both at the start and end of the chain. The only change to (4.16) is that $v^2 A_{n+1}$ is replaced by A_{n+1} . The solution (4.18) is the same. The double root condition now fixes $(n+1)/2$ of the constant terms appearing in the g_α for $\alpha = 1, \dots, n+1$, leaving us with the correct number of parameters to describe the Coulomb branch of $Sp(2k_1) \times SO(2k_2) \times Sp(2k_3) \times \cdots \times Sp(2k_n)$ with $(\mathbf{2k}_\alpha, \mathbf{2k}_{\alpha+1})$ matter together with d_α fundamentals in the α factor.

Product gauge groups of the form $SO(2k_1) \times Sp(2k_2) \times SO(2k_3) \times \cdots \times SO(2k_n)$ with fundamental matter

In this case we must have that $n+1$ is even. To obtain the curve in this case we replace the y^{n+1} term in (4.16) by $v^2 y^{n+1}$. The solution for the A_α (4.18) is identical. The double root condition now fixes $(n-1)/2$ of the constant terms appearing in the g_α for $\alpha = 1, \dots, n$. Thus we obtain the correct number of parameters to describe the Coulomb branch of $SO(2k_1) \times SO(2k_2) \times Sp(2k_3) \times \cdots \times SO(2k_n)$ with $(\mathbf{2k}_\alpha, \mathbf{2k}_{\alpha+1})$ matter and d_α fundamentals in the α factor.

Product gauge groups of the form $SO(2k_1+1) \times Sp(2k_2) \times SO(2k_3+1) \times \cdots \times SO(2k_n+1)$ with fundamental matter

When no semi-infinite fourbranes are present $n+1$ must be even in order that the curve admit a $v \rightarrow -v$, $t \rightarrow -t$ symmetry. The curve for this case takes the form

$$vy^{n+1} + A_1(v^2)y^n + \cdots + vA_{n+1}(v^2) = 0. \quad (4.20)$$

The same argument as above leads to the solution (4.18) for the A_α . The curve contains the expected number of parameters to describe the Coulomb branch of the theory.

5. Elliptic Models

In this section we will consider the generalization of the elliptic models of [14]. Namely, we compactify the coordinate x^6 on a circle of radius L . At certain points on this circle we place n fivebranes, and suspended between the $(\alpha-1)$ -th and the α -th fivebranes we have $2k_\alpha$ fourbranes. We identify the $\alpha=0$ fivebrane with the one at $\alpha=n$. An orientifold

plane will extend in the x^6 direction. Let us include also $2d_\alpha$ sixbranes localized between the $\alpha - 1$ -th and α -th fivebranes.

This configuration corresponds again to a product gauge group of alternating orthogonal and symplectic groups. With a compactified x^6 direction, consistency between flavor and gauge groups forces the number of fivebranes n to be even. We can attempt to describe chains with even, or chains with odd orthogonal gauge groups by setting the corresponding k_α to be respectively integer or half-integer.

An important property of the elliptic models for unitary gauge groups studied in [14] is that the product group included an additional $U(1)$ factor not present in the non-compact models, $G = U(1) \times \prod SU(k_\alpha)$. The origin of the extra $U(1)$ is that, for periodic configurations, equation (2.3) allows a global shift in the v plane of the brane configuration. In our case, even for elliptic models, this $U(1)$ is not present. It is always eliminated by the orientifold projection since it would correspond to moving the position of the orientifold plane and this is not a dynamical degree of freedom of the theory.

We are interested in models with non-positive β -functions. The sixbranes contribute a positive amount to the one-loop coefficient of the β -functions

$$b_{0,\alpha} = a_\alpha - a_{\alpha-1} + 2d_\alpha, \quad (5.1)$$

with a_α given by (3.23). If we want $b_{0,\alpha} \leq 0$ for each factor group G_α , we have to set $d_\alpha = 0$. Therefore the matter content of these theories will consist of half-hypermultiplets transforming in the $(\mathbf{2k}_\alpha, \mathbf{2k}_{\alpha+1})$ representations.

In the absence of sixbranes the condition $b_{0,\alpha} \leq 0$ reduces to $a_{\alpha-1} \geq a_\alpha$. This, together with the periodicity of our configuration, implies vanishing β -function coefficients and determines the number of fourbranes to be

$$k_\alpha = k - (1 - (-1)^\alpha) \frac{\omega_\alpha}{2}. \quad (5.2)$$

Let us fix $\omega_1 = 1$, the product group structure we get is

$$\cdots \times Sp(2k-2) \times SO(2k) \times Sp(2k-2) \times \cdots. \quad (5.3)$$

Only chains with even orthogonal groups can be derived from the elliptic models.

In order to obtain the exact solution of these models, as before, we lift our brane configuration to M -theory. The compact x^6 and x^{10} directions will define now a Riemann

surface of genus 1, which we denote by E . The elliptic curve E can generically be described in terms of two complex variables x, y by

$$y^2 = 4x^3 - g_2x - g_3, \quad (5.4)$$

with g_2, g_3 two complex numbers. The ambient space Q describing the directions x^4, x^5, x^6 and x^{10} is then the product $E \times C$. In [14] non-trivial fibrations of C over E were considered. They were associated with configurations periodic in x^6 up to a shift in the v coordinate. This shift translated into a bare mass parameter for the gauge theory on the fourbranes. Since it does not leave invariant the point $v = 0$, the introduction of a shift is not compatible with the orientifold projection. Therefore we will only consider the direct product space $Q = E \times C$. The absence of bare mass parameters for orthogonal and symplectic gauge theories where the flavor group has been gauged was already encountered in section (3.5).

Following [14], the Riemann surface solving the elliptic models will be described by a $2k$ -fold cover of the elliptic curve E

$$F(x, y, v^2) = v^{2k} + f_1(x, y)v^{2k-2} + \cdots + f_k(x, y) = 0, \quad (5.5)$$

with each branch related to positions of the fourbranes in the v plane. Let us analyze what is necessary for (5.5) to represent the answer. For each factor G_α in the product group, we denote by p_α the degree in v of the polynomial $F_{G_\alpha}(t, v^2)$ derived in section 3. In the case of $SO(2k_1)$ groups we have $p = 2k_1$ and for $Sp(2k_2)$ we have instead $p = 2k_2 + 2$. Equation (5.5) can provide the solution only when all the p_α in the cyclic chain are equal, in particular $p_\alpha = 2k$. This is precisely what we derive from the non-positivity of the β -functions.

The presence of fivebranes is encoded in (5.5) in terms of poles of the functions $f_i(x, y)$. More precisely, each function f_i will be chosen to have simple poles at n points p_1, \cdots, p_n representing the positions of the fivebranes.

To end this section we want to match the free parameters in (5.5) with parameters describing the moduli space of vacua of the gauge theory. The positions of the fivebranes provide n complex parameters which correspond to the n bare coupling constants of our finite theory. By the Riemann-Roch theorem, the space of meromorphic functions with n simple poles at p_1, \cdots, p_n on a Riemann surface of genus 1 is n -dimensional. Therefore (5.5) contains in addition nk parameters. The Coulomb branch of a gauge theory based on the product group (5.3) has dimension $nk - n/2$. Since we can not turn on bare masses

for the hypermultiplets, it seems that we get $n/2$ additional parameters without a gauge theory analogue.

This problem can be however solved by analyzing more carefully the behavior of (5.5) at $v = 0$. In sections (3.3) and (3.5) we saw that the strong interaction between the orientifold plane and the fivebranes gives rise to the condition that at $v = 0$ the fivebranes, pushed by the orientifold charge, should meet pairwise. In terms of (5.5) this implies that $f_k(x, y)$ should have $n/2$ double zeroes. A meromorphic function with n simple poles on a Riemann surface of genus 1 will have generically n simple zeroes. We have thus to impose that f_k has $n/2$ double zeroes. This represents $n/2$ constraints which eliminate $n/2$ parameters. Therefore the Riemann surface (5.5) contains exactly the number of parameters we need to represent the Casimirs of our gauge theory.

Finally let us comment on the Seiberg-Witten differential for the elliptic models. As in section 2, this is obtained by considering the area of a minimal volume membrane ending on the fivebrane. In this case the area is given by

$$V \sim \left| \int_D \frac{dv \, dx}{y} \right| = \left| \int_{\partial D} \frac{v \, dx}{y} \right|, \quad (5.6)$$

and the Seiberg-Witten differential is $\lambda = v \, dx/y$.

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